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On the Landau–Kallman–Rota Inequality

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1. In 1913 Edmund Landau proved an important inequality for twice differentiable functions which nowadays would be written

$$\| f' \|^2 \leqslant 4 \| f \| \| f'' \|, \tag{1.1}$$

where the norm refers to the space $C[0, \infty]$. It is also true for other spaces. In 1967 R. R. Kallman and G.-C. Rota [4] found the deeper reason for the inequality and showed that it could easily be derived from the general theory of contraction semigroups of linear transformations applied to the shift operator. This leads to a wide generalization of the inequality.

The connection is the following. Let X be a B-space over the complex numbers, $\{T(s)\}$ a semigroup of linear transformations from X into itself such that

$$\lim_{s \downarrow 0} T(s)\mathbf{x} = \mathbf{x}, \qquad \forall \mathbf{x}, \quad \text{and} \quad \| T(s) \| \leqslant 1, \quad \forall s > 0. \tag{1.2}$$

Let A be the infinitesimal generator of $\{T(s)\}$. Thus, for $\mathbf{x}$ in a linear subspace $D(A)$, dense in X,

$$\lim_{h \downarrow 0} \frac{1}{h} [T(h) - I]\mathbf{x} = A\mathbf{x}. \tag{1.3}$$

The domain of D^2 is also dense in X and, for $\mathbf{x} \in D(A^2)$, Taylor's theorem with remainder holds so that

$$T(s)\mathbf{x} = \mathbf{x} + \frac{s}{1!} A\mathbf{x} + \int_0^s (s - u)\, T(u)\, A^2\mathbf{x}\, du. \tag{1.4}$$

From this relation, Kallman and Rota derived the inequality

$$\| Ax \| \leqslant \frac{2}{s} \| \mathbf{x} \| + \frac{1}{2} s \| A^2\mathbf{x} \|. \tag{1.5}$$

If $A^2\mathbf{x} = 0$, it is seen that $A\mathbf{x} = 0$ and $\mathbf{x}$ is invariant under the operator $T(s)$ for all $s > 0$. If $A^2\mathbf{x} \neq 0$, it is seen that the right member of (1.5) becomes a minimum for

$$s = 2 \| \mathbf{x} \|^{\frac{1}{2}} \| A^2\mathbf{x} \|^{-\frac{1}{2}};$$

the minimum value gives

$$\| A\mathbf{x} \|^2 \leqslant 4 \| \mathbf{x} \| \| A^2\mathbf{x} \| \tag{1.6}$$

and this is the Landau–Kallman–Rota inequality.

Landau's theorem is obtained if X is a function space, say $C[0, \infty]$ or $L_p(0, \infty)$, and $T(s)$ is the shift operator defined by

$$T(s)[f](t) = f(s + t), \qquad s \geqslant 0; \tag{1.7}$$

for here

$$A[f](t) = f'(t). \tag{1.8}$$

2. The purpose of this note is to call attention to the inexhaustible supply of contraction semigroups. The corresponding infinitesimal generators are normally (always ?) differential operators so that among the inequalities (1.6) there is any number of direct generalizations of (1.1). Classical examples are the Gauss–Weierstrass and the Poisson semigroups. In the first case,

$$\omega(s)[f](t) = (\pi s)^{-\frac{1}{2}} \int_{\infty}^{\infty} \exp[-(t - u)^2/s] f(u)\, du, \tag{2.1}$$

with

$$A_\omega[f](t) = \tfrac{1}{4} f''(t) \tag{2.2}$$

and the inequality

$$\| f'' \|^2 \leqslant 4 \| f \| \| f^{(4)} \|. \tag{2.3}$$

In the Poisson case we have, instead,

$$P(s)[f](t) = \frac{s}{\pi} \int_{\infty}^{\infty} [(t - u)^2 + s^2]^{-1} f(u)\, du \tag{2.4}$$

with

$$A_p[f](t) = \tilde{f}'(t). \tag{2.5}$$

Here the tilde denotes the conjugate harmonic function. We note that $A_p{}^2 = 4A_\omega$ so that the LKR inequality gives

$$\| \tilde{f}' \|^2 \leqslant 4 \| f \| \| f'' \|.$$

In both cases X may be taken as $C[-\infty, \infty]$ or $L_p(-\infty, \infty)$, $1 \leqslant p \leqslant \infty$.

A more profitable way of attacking the problem is to verify that certain simple differential operators generate contraction semi-groups. The expressions for A_p and A_ω suggest trying

$$A_{2k-1}[f](t) = (-1)^{k-1} f^{(2k-1)}(t), \tag{2.8}$$
$$A_{2k}[f](t) = (-1)^{k-1} f^{(2k)}(t). \tag{2.9}$$

If the necessary number of derivatives are in $L_p(-\infty, \infty)$, $1 \leqslant p \leqslant 2$, then the Fourier transforms of the right members are

$$it^{2k-1}\hat{f}(t) \qquad \text{and} \qquad -t^{2k}\hat{f}(t), \tag{2.10}$$

respectively, where $\hat{f}$ is the Fourier transform of f. This leads to the factor transformations

$$U_{2k-1}(s)[\hat{f}](t) = \exp(it^{2k-1}s)\hat{f}(t), \qquad -\infty < s < \infty, \tag{2.11}$$
$$U_{2k}(s)[\hat{f}](t) = \exp(-t^{2k}s)\hat{f}(t), \qquad 0 \leqslant s. \tag{2.12}$$

The first is a group of isometries, the second a semigroup of contractions. Passing from Fourier transforms to functions, we get corresponding groups or semigroups of transformations $T_n(s)$. Here $T_n(s)[f]$ is the Fourier transform of $U_n(s)[\hat{f}]$. Furthermore, the infinitesimal generator of $T_n(s)$ is the corresponding operator A_n defined by (2.8) or (2.9). This leads to an infinite set of LKR inequalities generalizing the original Landau inequality. For all positive integers n,

$$\| f^{(n)} \|^2 \leqslant 4 \| f \| \, \| f^{(2n)} \|, \tag{2.13}$$

where $X = L_p(-\infty, \infty)$, $1 \leqslant p \leqslant 2$. While a proof based on Fourier transform theory does not apply for $2 < p$ or for C, the inequality would seem to be true also for these spaces.

It is of course possible to replace the single derivatives in (2.8) and (2.9) by suitably chosen differential polynomials with constant coefficients. Care must be taken that in the analog of (2.10) the multiplier has nonpositive real part. For such semigroups, see Hille–Phillips [2, pp. 574–580].

3. We can go much further and consider functions of several variables and corresponding operators involving partial derivatives. For the following, see Hille [1, pp. 400–408], not in Hille–Phillips [2]. We consider the space $L_2(R^m)$ and let $\mathbf{u} = (u_1, \ldots, u_m)$. If m is odd, $m = 2k - 1$, we form the scalar polynomial

$$A_{2k-1}(\mathbf{u}) = \sum_{j=0}^{k-1} (-1)^j P_{2j+1}(\mathbf{u}) \tag{3.1}$$

where each P_α is a homogeneous polynomial of degree α with real coefficients. For m even, $m = 2k$; set, instead,

$$\Lambda_{2k}(\mathbf{u}) = \sum_{j=0}^{k} (-1)^{j+1} P_{2j}(\mathbf{u}). \tag{3.2}$$

Here, again, each P_α is a homogeneous polynomial in $u_1, \ldots, u_m$ of degree α with real coefficients. In addition, we demand that P_{2k} be a positive definite m-ic form and that

$$P_e(\mathbf{u}) = \sum_{j=0}^{k} P_{2j}(\mathbf{u}) \tag{3.3}$$

be nonnegative everywhere in R^m. We write

$$P_0(\mathbf{u}) = \sum_{j=1}^{k-1} P_{2j+1}(\mathbf{u}). \tag{3.4}$$

The differential operator is now

$$\Lambda_m(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_m). \tag{3.5}$$

In the first case the Fourier transform of the result of applying the operator to f is

$$iP_0(\mathbf{t}) \hat{f}(\mathbf{t}), \tag{3.6}$$

and in the second case

$$-P_e(\mathbf{t}) \hat{f}(\mathbf{t}). \tag{3.7}$$

Here $\hat{f}$ is the Fourier transform of f. The formulas are valid on that subspace of $L_2(R^m)$ where the elements possess partial derivatives of the required orders and the partials belong to the space.

We have now corresponding factor semigroups on $L_2(R^m)$, namely,

$$U_{2k-1}(s)[\hat{f}](\mathbf{t}) = \exp[iP_0(\mathbf{t})s] \hat{f}(\mathbf{t}), \tag{3.8}$$

$$U_{2k}(s)[\hat{f}](\mathbf{t}) = \exp[-P_e(\mathbf{t})s] \hat{f}(\mathbf{t}), \tag{3.9}$$

respectively. The first transformation actually defines a group of isometries, $-\infty < s < \infty$, on the space into itself. The second set of transformations are contractions for $0 \leqslant s$. Passing now from Fourier transforms to functions we get a set of transformations $\{T_m(s)\}$. Here the Fourier transform of $T_m(s)[f]$ equals $U_m(s)[\hat{f}]$. Further, $\{T_m(s)\}$ is, for fixed m, a contraction semigroup (group if m is odd) and its infinitesimal generator is given by (3.5).

This leads to another family of LKR inequalities

$$\|\Lambda_m f\|^2 \leqslant 4\,\|f\|\,\|\Lambda_m^2 f\|. \tag{3.10}$$

4· We desist from further special cases but raise the question whether 4 is the best constant in all these inequalities. If we consider the class of all LKR inequalities (1.6), then this is certainly the case. This follows from the fact that 4 is needed in (1.1) for a special choice of X. But for other spaces a smaller value will do, already in the Landau case.

Now it is clear that in (2.13) we cannot replace the value 4 by anything smaller than 1 if the space under consideration contains e^{-t} as an element for which we have equality when 4 is replaced by 1. For the case $n = 2$ we can replace 4 by 4/3. This was observed by Kurepa [5] who got the result from his theory of cosine transforms. An elementary proof can be based on the identity

$$\begin{aligned}&\frac{1}{2}[f(t+s)+f(t-s)]\\ &\quad = f(t) + \frac{1}{2}s^2 f''(t) + \frac{1}{3!}\int_0^s (s-u)^3 \frac{1}{2}[f^{(4)}(u+t)+f^{(4)}(-u+t)]\,du \end{aligned}\tag{4.1}$$

valid for functions $t \to f(t)$ which together with their four first derivatives belong to the space for all t, $-\infty < t < \infty$. This gives the inequality

$$\|f''\| \leqslant 4s^{-2}\|f\| + \frac{1}{12}s^2\|f^{(4)}\|. \tag{4.2}$$

If $\|f^{(4)}\| \neq 0$ we can minimize the right member and obtain Kurepa's inequality

$$\|f''\|^2 \leqslant \frac{4}{3}\|f\|\,\|f^{(4)}\|. \tag{4.3}$$

This suggests that 4 can be replaced by a smaller number in (2.13) also for $n > 2$. The number should depend on n but what numbers are admissible is an open question.

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